Journal of Mathematical Sciences: Advances and Applications
Volume 7, Number 1, 2011, Pages 37-56

MATHEMATICAL INVERSE PROBLEM OF ELECTRIC POTENTIAL IN A HETEROGENEOUS LAYERED EARTH CONTAINING BURIED ELECTRODES

WARIN SRIPANYA[1] and SUABSAGUN YOOYUANYONG[2,3]

[1]Graduate School
Silpakorn University
Nakhon Pathom 73000
Thailand
e-mail: w.sripanya@windowslive.com

[2]Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom 73000
Thailand

[3]Centre of Excellence in Mathematics
Commission on Higher Education
Si Ayutthaya Rd.
Bangkok 10400
Thailand

Abstract

We derive analytical solutions of the electric potential resulting from a direct current point source located anywhere within two types of multilayered earth structures including layers having linearly varying conductivities and layers having binomially varying conductivities. Our solutions are obtained by solving a

2010 Mathematics Subject Classification: 86A20.

Keywords and phrases: Hankel transform, inverse problem, electric potential, buried current source.

Received December 30, 2010

boundary value problem in the wave number domain and then transforming the solution back to the spatial domain. The propagator matrix technique is used to formulate the upward-downward recurrences for solving the problems. One of these recurrences is applicable to general cases in which, the layers have constant, linearly or binomially varying conductivities. The equations derived for the electric potential can be used to interpret the hole-to-hole, hole-to-surface, and conventional surface array data. The inverse problems via the use of the Newton-Raphson and quasi-Newton optimization techniques are introduced for finding the conductivity parameters.

1. Introduction

In geophysical explorations, the traditional resistivity method maps the electrical properties of the earth by measuring differences in potential caused by a direct current flow between two current electrodes at the earth's surface. Usually interpretations of electrical soundings are conducted by assuming that the earth's structure consists of horizontally stratified layers having constant conductivities. A layered earth model is used to simulate the stratigraphic target. However, in the real earth situation, there are cases, where the subsurface conductivity varies continuously rather than discontinuously with depth. A particularly interesting case is a multilayered earth with one or more layers having linearly varying conductivities. The transitional layer can stand for the weathered zone in hard rock areas, where the degree of weathering diminishes with depth. This problem was first treated by Mallick and Roy [10], who obtained a theoretical solution for the problem of a two layered earth with transitional boundary. Jain [8] presented expressions for apparent resistivity of a three-layered earth, where the conductivity in the second layer varies linearly with depth and changes abruptly at the boundaries. Koefoed [9] solved the problem of a layered earth model containing an arbitrary number of homogeneous layers and of transitional layers in which the resistivities vary linearly with depth. Banerjee et al. [3] obtained expressions for apparent resistivity of a multilayered earth with a layer having binomially varying conductivity. All these investigations located a current source at the earth's surface.

The electrical method using buried electrodes is proposed for determining the apparent resistivity of multiple layers located underground. This particular electrode configuration is very useful when conventional electrical methods cannot be used, especially if the ground depth becomes very important. The calculated apparent resistivity shows a substantial quality increase in the measured signal caused by the underground targets, from which little or no response is measured by using conventional surface electrode methods. Investigations using buried electrodes were reported for layers having constant or exponentially varying conductivities. Alfano [1] considered a three-layered earth model with homogeneous layers and demonstrated that the uncertainty in the interpretation of resistivity soundings can be reduced by using buried electrodes. Daniels [7] presented a solution for the problem of a horizontally stratified layered earth containing an arbitrary number of layers having constant resistivities. Baumgartner [4] used submerged electrodes for determining the apparent resistivity of multiple homogeneous layers located underwater. Sato [12] developed a solution for the problem of a layered earth with all layers possessing exponentially varying resistivities.

In this study, we derive analytical solutions of the electric potential resulting from a direct current point source located anywhere within two types of multilayered earth structures including layers having linearly varying conductivities and layers having binomially varying conductivities. The Hankel transform is introduced to our problems and analytical results are derived. The inversion processes, using the Newton-Raphson and quasi-Newton methods, are conducted to estimate the conductivity parameters.

2. Model and Basic Equations

A geometric model of the earth's structure consists of two conductive half-spaces (see Figure 1). The half-space above the ground surface $(z < 0)$ is a region of air, whereas the half-space below the ground surface $(z > 0)$ is an n-layered horizontally stratified earth with depths to the layers $h_1, h_2, \ldots, h_{n-1}$ (the lowermost layer extending to infinity) measured from the ground surface, where $n \geq 2$ is an integer.

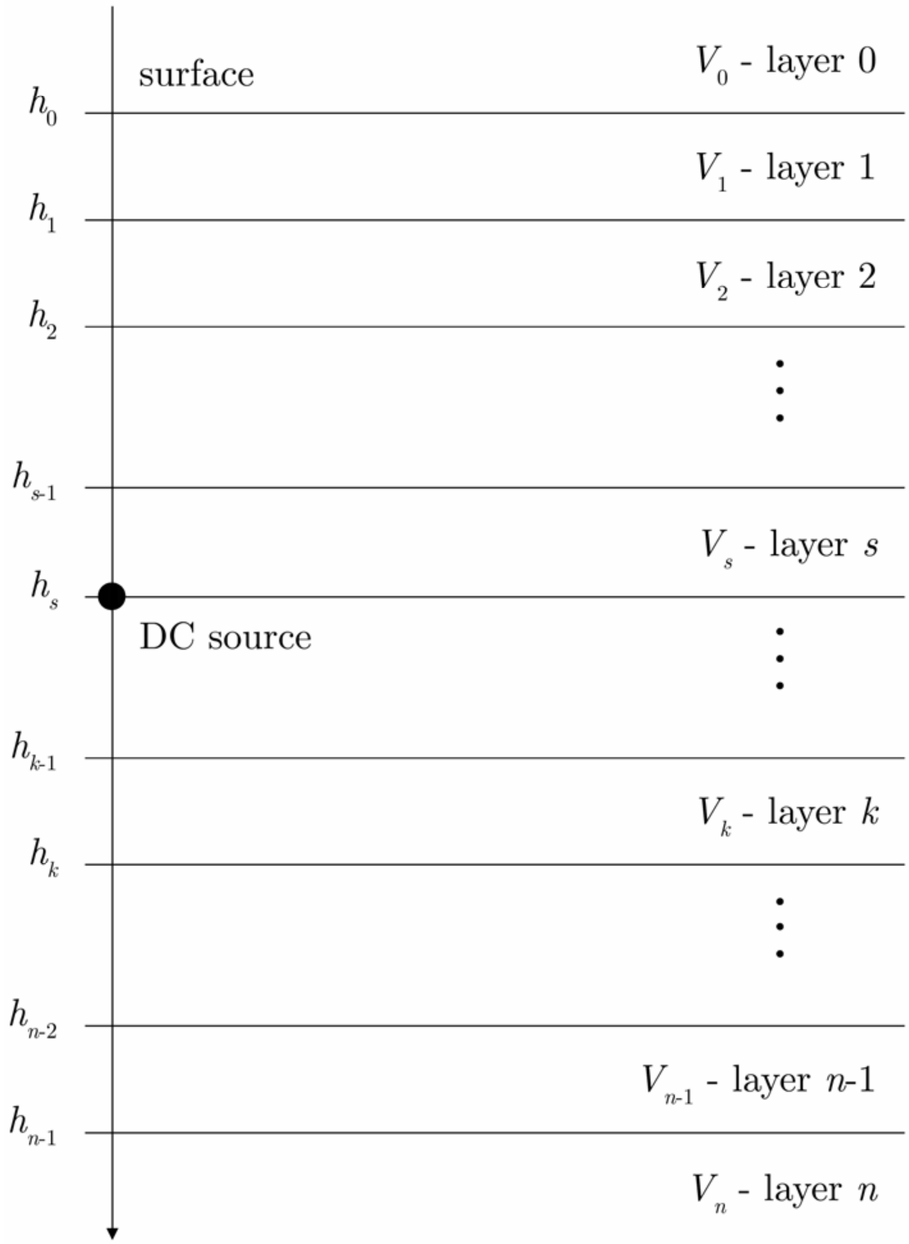

Figure 1. Geometric model of the earth's structure.

A point source of direct current I is deliberately located at the interface $z = h_s$ of layer s and layer $s+1 (1 \leq s \leq n-1)$ for simplifying the mathematics. Each layer has conductivity as a function of depth, i.e., $\sigma_k(z)$ for layer $0 \leq k \leq n.$ The electric potential V in direct current conditions satisfies the equation

$$\mathbf{E} = -\nabla V, \tag{1}$$

where $\mathbf{E}$ is the vector electric field. The vector current density $\mathbf{J}$ and the vector electric field are related through Ohm's law as

$$\mathbf{J} = \sigma \mathbf{E}, \tag{2}$$

where σ is the conductivity of the medium. The vector current density satisfies the equation:

$$\nabla \cdot \mathbf{J} = 0, \tag{3}$$

except at current sources or sinks. Since the problem is axi-symmetric in cylindrical coordinates $(r, \phi, z),$ it follows that V depends only on r and z. Eliminating $\mathbf{E}$ and $\mathbf{J}$ from Equations (1), (2), and (3), we obtain Poisson's equation in cylindrical coordinates

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial z}\frac{\partial V}{\partial z} = 0. \tag{4}$$

The Hankel transform (Ali and Kalla [2]) is introduced and defined by

$$\widetilde{V}(\lambda, z) = \int_0^\infty \lambda r V(r, z) J_0(\lambda r) dr, \tag{5}$$

and

$$V(r, z) = \int_0^\infty \widetilde{V}(\lambda, z) J_0(\lambda r) d\lambda, \tag{6}$$

where J_0 is the Bessel function of the first kind of order zero. Taking the transformation on both sides of Equation (4), we obtain

$$\frac{\partial^2 \widetilde{V}}{\partial z^2} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial z}\frac{\partial \widetilde{V}}{\partial z} - \lambda^2 \widetilde{V} = 0. \tag{7}$$

Therefore, the electric potential in each layer can be obtained by taking the inverse Hankel transform of the solution of Equation (7), which satisfies the following boundary conditions (Sato [12]):

(1) The electric potential V_0 tends to zero as z tends to minus infinity.

(2) The electric potential V_n tends to zero as z tends to infinity.

(3) The electric potential needs to be continuous on each of the boundary planes, i.e., for each $0 \le k \le n-1$,

$$\lim_{z \to h_k^-} \widetilde{V}_k = \lim_{z \to h_k^+} \widetilde{V}_{k+1}. \tag{8}$$

(4) The vertical component of the current density needs to be continuous on each of the boundary planes except on $z = h_s$, i.e., for each $0 \le k \le n-1$ and $k \ne s$,

$$\lim_{z \to h_k^-} \sigma_k \frac{\partial \widetilde{V}_k}{\partial z} = \lim_{z \to h_k^+} \sigma_{k+1} \frac{\partial \widetilde{V}_{k+1}}{\partial z}. \tag{9}$$

(5) The total current flowing out of any cylindrical surface around the current source must be equal to the current intensity, i.e., for any radius ξ,

$$\lim_{h \to 0}\left(\int_{\phi=0}^{2\pi} \int_{r=0}^{\xi} \mathbf{J}_s \big|_{z=h_s-h} \cdot (-\hat{z} r dr d\phi) \right.$$
$$\left. + \int_{\phi=0}^{2\pi} \int_{r=0}^{\xi} \mathbf{J}_{s+1} \big|_{z=h_s+h} \cdot (\hat{z} r dr d\phi) \right) = I. \tag{10}$$

3. Response of an Air Region

The equation for the electric potential in an air region, denoted by $\widetilde{V}_0$, can be determined by simplifying Equation (7) with constant σ_0 as

$$\frac{\partial^2 \widetilde{V}_0}{\partial z^2} - \lambda^2 \widetilde{V}_0 = 0, \tag{11}$$

and the solution is

$$\widetilde{V}_0(\lambda, z) = A_0 e^{\lambda z} + B_0 e^{-\lambda z}, \tag{12}$$

where A_0 and B_0 are arbitrary constants, which can be determined by using the boundary conditions. Thus, the electric potential in an air region is

$$V_0(r, z) = \int_0^\infty \left(A_0 e^{\lambda z} + B_0 e^{-\lambda z}\right) J_0(\lambda r) d\lambda. \tag{13}$$

4. Response of a Multilayered Earth with Layers Having Linearly Varying Conductivities

For each layer k, where $1 \le k \le n$ and $n \ge 2$, the variation of conductivity is denoted by

$$\sigma_k(z) = m_k z + c_k, \tag{14}$$

where c_k and $m_k \neq 0$ are constants, which preserve $\sigma_k(z) > 0$. Hence, the equation for the electric potential in layer k can be simplified by substituting Equation (14) into (7) as

$$\frac{\partial^2 \widetilde{V}_k}{\partial z^2} + \frac{m_k}{m_k z + c_k} \frac{\partial \widetilde{V}_k}{\partial z} - \lambda^2 \widetilde{V}_k = 0, \tag{15}$$

and the solution is

$$\widetilde{V}_k(\lambda, z) = A_k I_0\left(\frac{\lambda}{\varrho_k} \psi_k(z)\right) + B_k K_0\left(\frac{\lambda}{\varrho_k} \psi_k(z)\right), \tag{16}$$

where

$$\varrho_k = \frac{m_k}{c_k}, \qquad \psi_k(z) = 1 + \varrho_k z,$$

I_ν and K_ν are the modified Bessel functions of the first and second kinds of order ν, A_k and B_k are arbitrary constants, which can be determined by using the boundary conditions. Thus, the electric potential in layer k is

$$V_k(r, z) = \int_0^\infty \left(A_k I_0\left(\frac{\lambda}{\varrho_k} \psi_k(z) \right) + B_k K_0\left(\frac{\lambda}{\varrho_k} \psi_k(z) \right) \right) J_0(\lambda r) d\lambda. \tag{17}$$

Using the boundary conditions (8) and (9), for $0 \le k \le s-1$, A_{k+1} and B_{k+1} can be written in terms of A_k and B_k as

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = \Gamma_{k+1} \begin{bmatrix} A_k \\ B_k \end{bmatrix}, \tag{18}$$

where the propagator matrix is

$$\Gamma_{k+1} = \begin{bmatrix} M_{k+1}^{(11)} & M_{k+1}^{(12)} \\ M_{k+1}^{(21)} & M_{k+1}^{(22)} \end{bmatrix},$$

and for $1 \le k \le s-1$,

$$M_{k+1}^{(11)} = \frac{\lambda}{m_{k+1}} \left(\sigma_k^{\circ} I_1\left(\frac{\lambda}{\varrho_k} \psi_k(h_k) \right) K_0\left(\frac{\lambda}{\varrho_{k+1}} \psi_{k+1}(h_k) \right) + \sigma_{k+1}^{\bullet} I_0\left(\frac{\lambda}{\varrho_k} \psi_k(h_k) \right) K_1\left(\frac{\lambda}{\varrho_{k+1}} \psi_{k+1}(h_k) \right) \right),$$

$$M_{k+1}^{(12)} = \frac{\lambda}{m_{k+1}} \left(\sigma_{k+1}^{\bullet} K_0\left(\frac{\lambda}{\varrho_k} \psi_k(h_k) \right) K_1\left(\frac{\lambda}{\varrho_{k+1}} \psi_{k+1}(h_k) \right) - \sigma_k^{\circ} K_1\left(\frac{\lambda}{\varrho_k} \psi_k(h_k) \right) K_0\left(\frac{\lambda}{\varrho_{k+1}} \psi_{k+1}(h_k) \right) \right),$$

$$M_{k+1}^{(21)} = \frac{\lambda}{m_{k+1}} \left(\sigma_{k+1}^{\bullet} I_0\left(\frac{\lambda}{\varrho_k} \psi_k(h_k) \right) I_1\left(\frac{\lambda}{\varrho_{k+1}} \psi_{k+1}(h_k) \right) - \sigma_k^{\circ} I_1\left(\frac{\lambda}{\varrho_k} \psi_k(h_k) \right) I_0\left(\frac{\lambda}{\varrho_{k+1}} \psi_{k+1}(h_k) \right) \right),$$

$$M_{k+1}^{(22)} = \frac{\lambda}{m_{k+1}} \left(\sigma_k^{\circ} K_1\left(\frac{\lambda}{\varrho_k} \psi_k(h_k) \right) I_0\left(\frac{\lambda}{\varrho_{k+1}} \psi_{k+1}(h_k) \right) + \sigma_{k+1}^{\bullet} K_0\left(\frac{\lambda}{\varrho_k} \psi_k(h_k) \right) I_1\left(\frac{\lambda}{\varrho_{k+1}} \psi_{k+1}(h_k) \right) \right),$$

whereas if $k = 0$,

$$M_{k+1}^{(11)} = \frac{\lambda}{m_{k+1}}\left(\sigma_k^{\circ} K_0\left(\frac{\lambda}{\varrho_{k+1}}\right) + \sigma_{k+1}^{\bullet} K_1\left(\frac{\lambda}{\varrho_{k+1}}\right)\right),$$

$$M_{k+1}^{(12)} = \frac{\lambda}{m_{k+1}}\left(\sigma_{k+1}^{\bullet} K_1\left(\frac{\lambda}{\varrho_{k+1}}\right) - \sigma_k^{\circ} K_0\left(\frac{\lambda}{\varrho_{k+1}}\right)\right),$$

$$M_{k+1}^{(21)} = \frac{\lambda}{m_{k+1}}\left(\sigma_{k+1}^{\bullet} I_1\left(\frac{\lambda}{\varrho_{k+1}}\right) - \sigma_k^{\circ} I_0\left(\frac{\lambda}{\varrho_{k+1}}\right)\right),$$

$$M_{k+1}^{(22)} = \frac{\lambda}{m_{k+1}}\left(\sigma_k^{\circ} I_0\left(\frac{\lambda}{\varrho_{k+1}}\right) + \sigma_{k+1}^{\bullet} I_1\left(\frac{\lambda}{\varrho_{k+1}}\right)\right),$$

and

$$\lim_{z \to h_k^-} \sigma_k(z) = \sigma_k^{\circ}, \qquad \lim_{z \to h_k^+} \sigma_{k+1}(z) = \sigma_{k+1}^{\bullet}.$$

Hence, Equation (18) can be applied to obtain:

$$\begin{bmatrix} A_s \\ B_s \end{bmatrix} = \prod_{j=s}^{1} \Gamma_j \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}. \tag{19}$$

Similarly, if $s < n - 1$, then A_k and B_k can be written in terms of A_{k+1} and B_{k+1} as

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = \Theta_{k+1} \begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix}, \tag{20}$$

for $s + 1 \le k \le n - 1$, and the propagator matrix is given by

$$\Theta_{k+1} = \begin{bmatrix} N_{k+1}^{(11)} & N_{k+1}^{(12)} \\ N_{k+1}^{(21)} & N_{k+1}^{(22)} \end{bmatrix}, \tag{21}$$

where

$$N_{k+1}^{(11)} = \frac{\lambda}{m_k}\left(\sigma_k^{\circ} K_1\left(\frac{\lambda}{\varrho_k}\psi_k(h_k)\right) I_0\left(\frac{\lambda}{\varrho_{k+1}}\psi_{k+1}(h_k)\right)\right.$$
$$\left. + \sigma_{k+1}^{\bullet} K_0\left(\frac{\lambda}{\varrho_k}\psi_k(h_k)\right) I_1\left(\frac{\lambda}{\varrho_{k+1}}\psi_{k+1}(h_k)\right)\right),$$

$$N_{k+1}^{(12)} = \frac{\lambda}{m_k}\left(\sigma_k^{\circ} K_1\left(\frac{\lambda}{\varrho_k}\psi_k(h_k)\right)K_0\left(\frac{\lambda}{\varrho_{k+1}}\psi_{k+1}(h_k)\right)\right.$$

$$\left. - \sigma_{k+1}^{\bullet} K_0\left(\frac{\lambda}{\varrho_k}\psi_k(h_k)\right)K_1\left(\frac{\lambda}{\varrho_{k+1}}\psi_{k+1}(h_k)\right)\right),$$

$$N_{k+1}^{(21)} = \frac{\lambda}{m_k}\left(\sigma_k^{\circ} I_1\left(\frac{\lambda}{\varrho_k}\psi_k(h_k)\right)I_0\left(\frac{\lambda}{\varrho_{k+1}}\psi_{k+1}(h_k)\right)\right.$$

$$\left. - \sigma_{k+1}^{\bullet} I_0\left(\frac{\lambda}{\varrho_k}\psi_k(h_k)\right)I_1\left(\frac{\lambda}{\varrho_{k+1}}\psi_{k+1}(h_k)\right)\right),$$

$$N_{k+1}^{(22)} = \frac{\lambda}{m_k}\left(\sigma_k^{\circ} I_1\left(\frac{\lambda}{\varrho_k}\psi_k(h_k)\right)K_0\left(\frac{\lambda}{\varrho_{k+1}}\psi_{k+1}(h_k)\right)\right.$$

$$\left. + \sigma_{k+1}^{\bullet} I_0\left(\frac{\lambda}{\varrho_k}\psi_k(h_k)\right)K_1\left(\frac{\lambda}{\varrho_{k+1}}\psi_{k+1}(h_k)\right)\right).$$

Hence, Equation (20) can also be applied to obtain:

$$\begin{bmatrix} A_{s+1} \\ B_{s+1} \end{bmatrix} = \Theta^* \begin{bmatrix} A_n \\ B_n \end{bmatrix}, \tag{22}$$

where

$$\Theta^* = \begin{cases} \displaystyle\prod_{j=s+2}^{n} \Theta_j, & \text{if } s < n-1, \\ \mathbf{I}_2, & \text{if } s = n-1. \end{cases}$$

On the boundary plane $z = h_s$, the boundary condition (10) requires

$$I = \int_0^{\infty} 2\pi\xi\left(\sigma_s^{\circ}\left(A_s I_1\left(\frac{\lambda}{\varrho_s}\psi_s(h_s)\right)\right.\right.$$

$$\left. - B_s K_1\left(\frac{\lambda}{\varrho_s}\psi_s(h_s)\right)\right) + \sigma_{s+1}^{\bullet}\left(B_{s+1} K_1\left(\frac{\lambda}{\varrho_{s+1}}\psi_{s+1}(h_s)\right)\right.$$

$$\left.\left. - A_{s+1} I_1\left(\frac{\lambda}{\varrho_{s+1}}\psi_{s+1}(h_s)\right)\right)\right) J_1(\lambda\xi)\, d\lambda. \tag{23}$$

Inverting Equation (23) by using the Fourier-Bessel integral (Watson [13]) yields

$$I = 2\pi\Bigg(\sigma_s^{\circ}\bigg(A_s I_1\Big(\frac{\lambda}{\varrho_s}\psi_s(h_s)\Big) - B_s K_1\Big(\frac{\lambda}{\varrho_s}\psi_s(h_s)\Big)\bigg)$$

$$+ \sigma_{s+1}^{\bullet}\bigg(B_{s+1} K_1\Big(\frac{\lambda}{\varrho_{s+1}}\psi_{s+1}(h_s)\Big) - A_{s+1} I_1\Big(\frac{\lambda}{\varrho_{s+1}}\psi_{s+1}(h_s)\Big)\bigg)\Bigg). \tag{24}$$

Thus, the boundary conditions (8) and (10) lead to

$$\begin{bmatrix} A_s \\ B_s \end{bmatrix} = \Theta_{s+1}\begin{bmatrix} A_{s+1} \\ B_{s+1} \end{bmatrix} + \frac{I\lambda}{2\pi m_s}\begin{bmatrix} K_0((\lambda/\varrho_s)\psi_s(h_s)) \\ -I_0((\lambda/\varrho_s)\psi_s(h_s)) \end{bmatrix}, \tag{25}$$

where Θ_{s+1} is determined by replacing k with s in Equation (21). Substituting Equations (19) and (22) into (25), we obtain

$$\prod_{j=s}^{1}\Gamma_j\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \Theta_{s+1}\Theta^{*}\begin{bmatrix} A_n \\ B_n \end{bmatrix} + \frac{I\lambda}{2\pi m_s}\begin{bmatrix} K_0((\lambda/\varrho_s)\psi_s(h_s)) \\ -I_0((\lambda/\varrho_s)\psi_s(h_s)) \end{bmatrix}. \tag{26}$$

To guarantee the convergence of the electric potential V_0 when z tends to minus infinity, we must take

$$B_0 = 0. \tag{27}$$

Similarly, the convergence of the electric potential V_n when z tends to infinity can be guaranteed by taking

$$A_n = 0. \tag{28}$$

Therefore, Equation (26) can be rewritten as a system of two linear equations in terms of the unknowns A_0 and B_n as

$$\begin{bmatrix} U_{11} & -V_{12} \\ U_{21} & -V_{22} \end{bmatrix}\begin{bmatrix} A_0 \\ B_n \end{bmatrix} = \frac{I\lambda}{2\pi m_s}\begin{bmatrix} K_0((\lambda/\varrho_s)\psi_s(h_s)) \\ -I_0((\lambda/\varrho_s)\psi_s(h_s)) \end{bmatrix}, \tag{29}$$

where U_{ij} and V_{ij} are determined by

$$\prod_{j=s}^{1}\Gamma_j = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \qquad \Theta_{s+1}\Theta^{*} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Since the coefficient matrix of this system is nonsingular (see Chen and Oldenburg [6], Yooyuanyong and Sripanya [15, 17]), by Cramer's rule, the system has a unique solution

$$A_0 = \frac{I\lambda}{2\pi m_s} \frac{V_{12} I_0((\lambda/\varrho_s)\psi_s(h_s)) + V_{22} K_0((\lambda/\varrho_s)\psi_s(h_s))}{U_{11}V_{22} - U_{21}V_{12}}, \tag{30}$$

$$B_n = \frac{I\lambda}{2\pi m_s} \frac{U_{11} I_0((\lambda/\varrho_s)\psi_s(h_s)) + U_{21} K_0((\lambda/\varrho_s)\psi_s(h_s))}{U_{11}V_{22} - U_{21}V_{12}}. \tag{31}$$

As A_0, B_0, A_n, and B_n are determined, so A_k and B_k, where $k \neq 0$ and n can be obtained from the upward-downward recurrence as shown in Equations (18) and (20).

5. Response of a Multilayered Earth with Layers Having Binomially Varying Conductivities

For each layer k, where $1 \leq k \leq n$ and $n \geq 2$, the variation of conductivity is denoted by

$$\sigma_k(z) = c_k(1 + d_k z)^{p_k}, \tag{32}$$

where c_k, p_k, and $d_k \neq 0$ are constants, which preserve $\sigma_k(z) > 0$. Hence, the equation for the electric potential in layer k can be simplified by substituting Equation (32) into (7) as

$$\frac{\partial^2 \widetilde{V}_k}{\partial z^2} + \frac{d_k p_k}{1 + d_k z} \frac{\partial \widetilde{V}_k}{\partial z} - \lambda^2 \widetilde{V}_k = 0, \tag{33}$$

and the solution is

$$\widetilde{V}_k(\lambda, z) = \widetilde{\psi}_k^{\gamma_k}(z)\left(A_k I_{-\gamma_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\psi}_k(z)\right) + B_k K_{-\gamma_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\psi}_k(z)\right)\right), \tag{34}$$

where

$$\widetilde{\varrho}_k = d_k, \qquad \widetilde{\psi}_k(z) = 1 + \widetilde{\varrho}_k z, \qquad \gamma_k = \frac{1 - p_k}{2},$$

A_k and B_k are arbitrary constants, which can be determined by using the boundary conditions. Thus, the electric potential in layer k is

$$V_k(r, z) = \int_0^\infty \tilde{\vartheta}_k^{\gamma_k}(z)\left(A_k I_{-\gamma_k}\left(\frac{\lambda}{\tilde{\varrho}_k}\tilde{\vartheta}_k(z)\right)\right.$$

$$\left. + B_k K_{-\gamma_k}\left(\frac{\lambda}{\tilde{\varrho}_k}\tilde{\vartheta}_k(z)\right)\right) J_0(\lambda r) d\lambda. \tag{35}$$

Proceeding as in Section 4, after using the boundary conditions, we obtain

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = \tilde{\Gamma}_{k+1} \begin{bmatrix} A_k \\ B_k \end{bmatrix}, \tag{36}$$

for $0 \leq k \leq s-1$, and the propagator matrix is given by

$$\tilde{\Gamma}_{k+1} = \begin{bmatrix} \tilde{M}_{k+1}^{(11)} & \tilde{M}_{k+1}^{(12)} \\ \tilde{M}_{k+1}^{(21)} & \tilde{M}_{k+1}^{(22)} \end{bmatrix},$$

and for $1 \leq k \leq s-1$,

$$\tilde{M}_{k+1}^{(11)} = \frac{\lambda}{c_{k+1}d_{k+1}}\left(\sigma_k^{\circ} I_{\upsilon_k}\left(\frac{\lambda}{\tilde{\varrho}_k}\tilde{\vartheta}_k(h_k)\right) K_{\gamma_{k+1}}\left(\frac{\lambda}{\tilde{\varrho}_{k+1}}\tilde{\vartheta}_{k+1}(h_k)\right)\right.$$

$$\left. + \sigma_{k+1}^{\bullet} I_{\gamma_k}\left(\frac{\lambda}{\tilde{\varrho}_k}\tilde{\vartheta}_k(h_k)\right) K_{\upsilon_{k+1}}\left(\frac{\lambda}{\tilde{\varrho}_{k+1}}\tilde{\vartheta}_{k+1}(h_k)\right)\right)\tilde{\vartheta}_k^{\gamma_k}(h_k)\tilde{\vartheta}_{k+1}^{\gamma_{k+1}}(h_k),$$

$$\tilde{M}_{k+1}^{(12)} = \frac{\lambda}{c_{k+1}d_{k+1}}\left(\sigma_{k+1}^{\bullet} K_{\gamma_k}\left(\frac{\lambda}{\tilde{\varrho}_k}\tilde{\vartheta}_k(h_k)\right) K_{\upsilon_{k+1}}\left(\frac{\lambda}{\tilde{\varrho}_{k+1}}\tilde{\vartheta}_{k+1}(h_k)\right)\right.$$

$$\left. - \sigma_k^{\circ} K_{\upsilon_k}\left(\frac{\lambda}{\tilde{\varrho}_k}\tilde{\vartheta}_k(h_k)\right) K_{\gamma_{k+1}}\left(\frac{\lambda}{\tilde{\varrho}_{k+1}}\tilde{\vartheta}_{k+1}(h_k)\right)\right)\tilde{\vartheta}_k^{\gamma_k}(h_k)\tilde{\vartheta}_{k+1}^{\gamma_{k+1}}(h_k),$$

$$\tilde{M}_{k+1}^{(21)} = \frac{\lambda}{c_{k+1}d_{k+1}}\left(\sigma_{k+1}^{\bullet} I_{\gamma_k}\left(\frac{\lambda}{\tilde{\varrho}_k}\tilde{\vartheta}_k(h_k)\right) I_{\upsilon_{k+1}}\left(\frac{\lambda}{\tilde{\varrho}_{k+1}}\tilde{\vartheta}_{k+1}(h_k)\right)\right.$$

$$\left. - \sigma_k^{\circ} I_{\upsilon_k}\left(\frac{\lambda}{\tilde{\varrho}_k}\tilde{\vartheta}_k(h_k)\right) I_{\gamma_{k+1}}\left(\frac{\lambda}{\tilde{\varrho}_{k+1}}\tilde{\vartheta}_{k+1}(h_k)\right)\right)\tilde{\vartheta}_k^{\gamma_k}(h_k)\tilde{\vartheta}_{k+1}^{\gamma_{k+1}}(h_k),$$

$$\widetilde{M}_{k+1}^{(22)} = \frac{\lambda}{c_{k+1}d_{k+1}}\Bigg(\sigma_k^{\circ}K_{\upsilon_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\psi}_k(h_k)\right)I_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\psi}_{k+1}(h_k)\right)$$

$$+\,\sigma_{k+1}^{\bullet}K_{\gamma_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\psi}_k(h_k)\right)I_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\psi}_{k+1}(h_k)\right)\Bigg)\widetilde{\psi}_k^{\gamma_k}(h_k)\widetilde{\psi}_{k+1}^{\gamma_{k+1}}(h_k),$$

whereas if $k = 0$,

$$\widetilde{M}_{k+1}^{(11)} = \frac{\lambda}{c_{k+1}d_{k+1}}\left(\sigma_k^{\circ}K_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\right) + \sigma_{k+1}^{\bullet}K_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\right)\right),$$

$$\widetilde{M}_{k+1}^{(12)} = \frac{\lambda}{c_{k+1}d_{k+1}}\left(\sigma_{k+1}^{\bullet}K_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\right) - \sigma_k^{\circ}K_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\right)\right),$$

$$\widetilde{M}_{k+1}^{(21)} = \frac{\lambda}{c_{k+1}d_{k+1}}\left(\sigma_{k+1}^{\bullet}I_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\right) - \sigma_k^{\circ}I_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\right)\right),$$

$$\widetilde{M}_{k+1}^{(22)} = \frac{\lambda}{c_{k+1}d_{k+1}}\left(\sigma_k^{\circ}I_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\right) + \sigma_{k+1}^{\bullet}I_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\right)\right),$$

and

$$\upsilon_k = \frac{1 + p_k}{2}.$$

Similarly, if $s < n - 1$, we have

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = \widetilde{\Theta}_{k+1}\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix}, \tag{37}$$

for $s + 1 \le k \le n - 1$, and we also have

$$\widetilde{\Theta}_{k+1} = \begin{bmatrix} \widetilde{N}_{k+1}^{(11)} & \widetilde{N}_{k+1}^{(12)} \\ \widetilde{N}_{k+1}^{(21)} & \widetilde{N}_{k+1}^{(22)} \end{bmatrix}, \tag{38}$$

where

$$\widetilde{N}_{k+1}^{(11)} = \frac{\lambda}{c_k d_k}\Bigg(\sigma_k^{\circ}K_{\upsilon_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\psi}_k(h_k)\right)I_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\psi}_{k+1}(h_k)\right)$$

$$+\,\sigma_{k+1}^{\bullet}K_{\gamma_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\psi}_k(h_k)\right)I_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\psi}_{k+1}(h_k)\right)\Bigg)\widetilde{\psi}_k^{\gamma_k}(h_k)\widetilde{\psi}_{k+1}^{\gamma_{k+1}}(h_k),$$

$$\widetilde{N}_{k+1}^{(12)} = \frac{\lambda}{c_k d_k}\left(\sigma_k^{\circ} K_{\upsilon_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\vartheta}_k(h_k)\right)K_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\vartheta}_{k+1}(h_k)\right)\right.$$

$$\left. - \sigma_{k+1}^{\bullet} K_{\gamma_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\vartheta}_k(h_k)\right)K_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\vartheta}_{k+1}(h_k)\right)\right)\widetilde{\vartheta}_k^{\gamma_k}(h_k)\widetilde{\vartheta}_{k+1}^{\gamma_{k+1}}(h_k),$$

$$\widetilde{N}_{k+1}^{(21)} = \frac{\lambda}{c_k d_k}\left(\sigma_k^{\circ} I_{\upsilon_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\vartheta}_k(h_k)\right)I_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\vartheta}_{k+1}(h_k)\right)\right.$$

$$\left. - \sigma_{k+1}^{\bullet} I_{\gamma_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\vartheta}_k(h_k)\right)I_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\vartheta}_{k+1}(h_k)\right)\right)\widetilde{\vartheta}_k^{\gamma_k}(h_k)\widetilde{\vartheta}_{k+1}^{\gamma_{k+1}}(h_k),$$

$$\widetilde{N}_{k+1}^{(22)} = \frac{\lambda}{c_k d_k}\left(\sigma_k^{\circ} I_{\upsilon_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\vartheta}_k(h_k)\right)K_{\gamma_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\vartheta}_{k+1}(h_k)\right)\right.$$

$$\left. + \sigma_{k+1}^{\bullet} I_{\gamma_k}\left(\frac{\lambda}{\widetilde{\varrho}_k}\widetilde{\vartheta}_k(h_k)\right)K_{\upsilon_{k+1}}\left(\frac{\lambda}{\widetilde{\varrho}_{k+1}}\widetilde{\vartheta}_{k+1}(h_k)\right)\right)\widetilde{\vartheta}_k^{\gamma_k}(h_k)\widetilde{\vartheta}_{k+1}^{\gamma_{k+1}}(h_k).$$

Hence, Equations (36) and (37) can be applied to obtain

$$\begin{bmatrix} A_s \\ B_s \end{bmatrix} = \prod_{j=s}^{1}\widetilde{\Gamma}_j\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \qquad \begin{bmatrix} A_{s+1} \\ B_{s+1} \end{bmatrix} = \widetilde{\Theta}^*\begin{bmatrix} A_n \\ B_n \end{bmatrix}, \tag{39}$$

where

$$\widetilde{\Theta}^* = \begin{cases} \prod_{j=s+2}^{n}\widetilde{\Theta}_j, & \text{if } s < n-1, \\ \mathbf{I}_2, & \text{if } s = n-1. \end{cases}$$

On the boundary plane $z = h_s$, it follows from Equation (10) that

$$\begin{bmatrix} A_s \\ B_s \end{bmatrix} = \widetilde{\Theta}_{s+1}\begin{bmatrix} A_{s+1} \\ B_{s+1} \end{bmatrix} + \widetilde{\vartheta}_s^{\gamma_s}(h_s)\frac{I\lambda}{2\pi c_s d_s}\begin{bmatrix} K_{\gamma_s}\left((\lambda/\widetilde{\varrho}_s)\widetilde{\vartheta}_s(h_s)\right) \\ -I_{\gamma_s}\left((\lambda/\widetilde{\varrho}_s)\widetilde{\vartheta}_s(h_s)\right) \end{bmatrix}, \tag{40}$$

where $\widetilde{\Theta}_{s+1}$ is determined by replacing k with s in Equation (38). Substituting Equation (39) into (40) and using

$$B_0 = 0 = A_n, \tag{41}$$

we obtain a system of two linear equations in terms of the unknowns A_0 and B_n as

$$\begin{bmatrix} \widetilde{U}_{11} & -\widetilde{V}_{12} \\ \widetilde{U}_{21} & -\widetilde{V}_{22} \end{bmatrix} \begin{bmatrix} A_0 \\ B_n \end{bmatrix} = \widetilde{\vartheta}_s^{\gamma_s}(h_s) \frac{I\lambda}{2\pi c_s d_s} \begin{bmatrix} K_{\gamma_s}\left((\lambda / \widetilde{\varrho}_s)\widetilde{\vartheta}_s(h_s)\right) \\ -I_{\gamma_s}\left((\lambda / \widetilde{\varrho}_s)\widetilde{\vartheta}_s(h_s)\right) \end{bmatrix}, \tag{42}$$

where $\widetilde{U}_{ij}$ and $\widetilde{V}_{ij}$ are determined by

$$\prod_{j=s}^{1} \widetilde{\Gamma}_j = \begin{bmatrix} \widetilde{U}_{11} & \widetilde{U}_{12} \\ \widetilde{U}_{21} & \widetilde{U}_{22} \end{bmatrix}, \qquad \widetilde{\Theta}_{s+1}\widetilde{\Theta}^* = \begin{bmatrix} \widetilde{V}_{11} & \widetilde{V}_{12} \\ \widetilde{V}_{21} & \widetilde{V}_{22} \end{bmatrix}.$$

Since the coefficient matrix of this system is nonsingular (see Chen and Oldenburg [6], Yooyuanyong and Sripanya [15, 17]), by Cramer's rule, the system has a unique solution

$$A_0 = \widetilde{\vartheta}_s^{\gamma_s}(h_s) \frac{I\lambda}{2\pi c_s d_s} \frac{\widetilde{V}_{12} I_{\gamma_s}\left((\lambda / \widetilde{\varrho}_s)\widetilde{\vartheta}_s(h_s)\right) + \widetilde{V}_{22} K_{\gamma_s}\left((\lambda / \widetilde{\varrho}_s)\widetilde{\vartheta}_s(h_s)\right)}{\widetilde{U}_{11}\widetilde{V}_{22} - \widetilde{U}_{21}\widetilde{V}_{12}},$$

$$B_n = \widetilde{\vartheta}_s^{\gamma_s}(h_s) \frac{I\lambda}{2\pi c_s d_s} \frac{\widetilde{U}_{11} I_{\gamma_s}\left((\lambda / \widetilde{\varrho}_s)\widetilde{\vartheta}_s(h_s)\right) + \widetilde{U}_{21} K_{\gamma_s}\left((\lambda / \widetilde{\varrho}_s)\widetilde{\vartheta}_s(h_s)\right)}{\widetilde{U}_{11}\widetilde{V}_{22} - \widetilde{U}_{21}\widetilde{V}_{12}}.$$

As A_0, B_0, A_n, and B_n are determined, so A_k and B_k, where $k \neq 0$ and n, can be obtained from the upward-downward recurrence as shown in Equations (36) and (37).

6. Numerical Experiments and Inversion Processes

In our inverse model examples, we simulate array data of the electric potential from two forward models of the earth structures. Both of the example ground models have two layers. The buried depth of current source for our entire models is 10 meters. The conductivity in the region of air is approximately equal to zero. The overburden has a constant conductivity denoted by a with thickness h, whereas the host has a linearly varying conductivity denoted by $\sigma(z) = a + m(z - h)$ with infinite depth. The values of model parameters are given in Table 1.

Table 1. Model parameters used in our inversion examples

Model	a (in S/m)	m (in S/m^2)	h (in m)
1	0.1692857143	0.0261904761	10
2	0.1692857143	0.0261904761	15

The parameter a is a conductivity at the ground surface, which can be assumed to be known from the measurement. The iterative procedure using the Newton-Raphson method is applied to estimate the unknown parameter m of the first example model, whereas the unknown parameters m and h of the second model are estimated by using the quasi-Newton optimization technique. Chave's algorithm [5] is used for numerically calculating the inverse Hankel transform of the electric potential solutions. The special functions are computed by using the numerical recipes source codes (Press et al. [11]). Random errors up to 3% are superimposed on the scaled electric potential from the forward problem to simulate the set of real data. We start the iterative processes to find the values of the conductivity parameters with initial guess $m = 0.01\,\mathrm{S/m^2}$ and $h = 10\mathrm{m}$. The optimal result of the first model converges to the true value with percentage error less than 2.3% after using 19 iterations. The graph of the true and calculated conductivity models are plotted as shown in Figure 2.

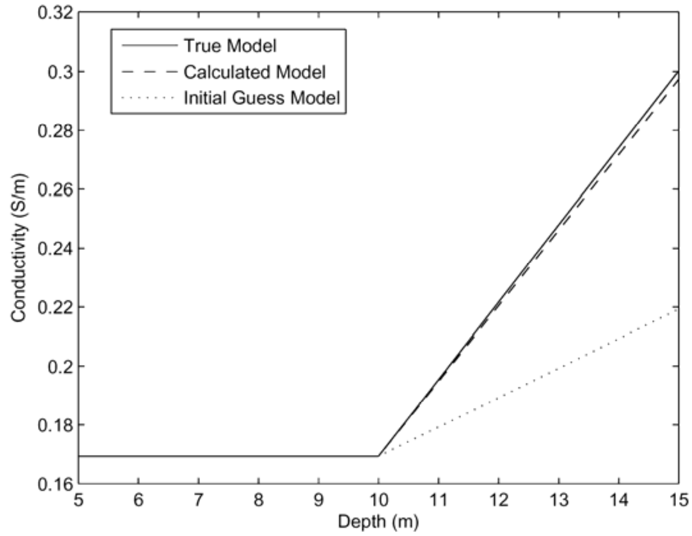

Figure 2. Graph of conductivity σ against depth z for our first model.

We see that the graph of the calculated model is closed to the true model. The inversion of the second model leads to the best value having percentage errors about 4.3% and 8.0% for the parameters h and m, respectively, after using 23 iterations. Figure 3 shows the true and calculated conductivity models for the second model example. The graph of the calculated model is also closed to the true model. These illustrate the advantage in using the Newton-Raphson and quasi-Newton methods, which give the result much better than using another method of inversion, especially for estimating only one parameter (e.g., Yooyuanyong [14, 16]). We note that the number of unknown parameters in the first model example is less than the second example. Not surprisingly, the convergence of inversion for the first model is faster than the second model. Moreover, the optimal result of the first model is also better than the second model.

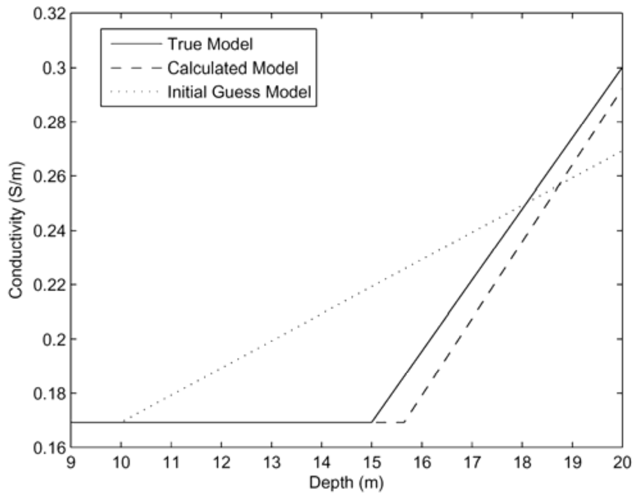

Figure 3. Graph of conductivity σ against depth z for our second model.

7. Conclusion

We derive analytical solutions of the electric potential resulting from a buried current source by using the recurrence relations. The solutions can be used to interpret the hole-to-hole, hole-to-surface, and conventional surface array data (the buried depth of current source is assumed to be zero). Two simple cases are used to investigate the conductivity profiles. The inversion processes, using the Newton-Raphson and quasi-Newton methods, are conducted to estimate the conductivity parameters. The methods lead to very good results and have the robustness of the procedures.

Acknowledgement

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

References

[1] L. Alfano, Geoelectrical prospecting with underground electrodes, Geophys. Prosp. 10(3) (1962), 290-303.

[2] I. Ali and S. Kalla, A generalized Hankel transform and its use for solving certain partial differential equations, J. Austral. Math. Soc. Ser. B 41(1) (1999), 105-117.

[3] B. Banerjee, B. J. Sengupta and B. P. Pal, Resistivity sounding on a multilayered earth containing transition layers, Geophys. Prosp. 28(5) (1980), 750-758.

[4] F. Baumgartner, A new method for geoelectrical investigations underwater, Geophys. Prosp. 44(1) (1996), 71-98.

[5] A. D. Chave, Numerical integration of related Hankel transforms by quadrature and continued fraction expansion, Geophysics 48(12) (1983), 1671-1686.

[6] J. Chen and D. W. Oldenburg, Magnetic and electrical fields of direct currents in a layered earth, Expl. Geophys. 35(2) (2004), 157-163.

[7] J. J. Daniels, Interpretation of buried electrode resistivity data using a layered earth model, Geophysics 43(5) (1978), 988-1001.

[8] S. C. Jain, Resistivity sounding on a three-layer transitional model, Geophys. Prosp. 20(2) (1972), 283-292.

[9] O. Koefoed, Resistivity sounding on an earth model containing transition layers with linear change of resistivity with depth, Geophys. Prosp. 27(4) (1979), 862-868.

[10] K. Mallick and A. Roy, Resistivity sounding on a two-layer earth with transitional boundary, Geophys. Prosp. 16(4) (1968), 436-446.

[11] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, Numerical Recipes in Fortran 77: The Art of Scientific Computing, Fortran Numerical Recipes Vol. 1, 2nd ed., Cambridge Univ. Press, 1992.

[12] H. K. Sato, Potential field from a dc current source arbitrarily located in a nonuniform layered medium, Geophysics 65(6) (2000), 1726-1732.

[13] G. N. Watson, A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge Univ. Press, 1995.

[14] S. Yooyuanyong, Mathematical model of MMR inversion for geophysical data, Songklanakarin J. Sci. Technol. 29(4) (2007), 1417-1426.

[15] S. Yooyuanyong and W. Sripanya, Analytical solutions of magnetic field response from multilayered earth structures, Y. A. Hasan, N. M. Ali and A. I. M. Ismail, eds., Proceedings of the 5th Asian Mathematical Conference, Vol. II: Applied Mathematics (2009), 411-418.

[16] S. Yooyuanyong, Mathematical inverse problem of electric and vertical magnetic fields to resolve saline soil structure, J. Math. Sci. Adv. Appl. (in press).

[17] S. Yooyuanyong and W. Sripanya, Inversion of magnetic field for transitional ground profile, Thai. J. Math. (submitted).

■